TROMSØ FORSKNINGSSTIFTELSE

FINAL REPORT – deadline within 30 days of project completion

The report may be in Norwegian or English.

I. PROJECT INFORMATION		
Project title:	*Symmetry in Quantitative and Algorithmic Real Algebraic Geometry*	
TFS project ID:	*17MatCR*	
UiT project ID:	*SymRAG*	
Principle investigator:	*Cordian Riener*	
Position now:	*Professor*	
Is the position dependent on external funding?	Yes ☐	No ☒
Project host:	Department for Mathematics and Statistics (IMS)	
Project start date	Project end date	Status report date
01.03.2018	31.12.2024	28.02.2025
Foundation funding	Own funding (UiT)	Other funding <*specify source*>
6,3 + 4 MNOK	8,5+6 MNOK	14 MNOK (EU- see report for details)

II. PROJECT PURPOSE

The primary purpose of this project was to advance our understanding of symmetry in both qualitative and algorithmic contexts within real algebraic geometry. By exploring how the invariance properties of algebraic structures — such as polynomial systems — can be harnessed, the project aimed to develop new computational methods and theoretical insights that simplify complex mathematical problems. Central to this endeavor was the examination of how symmetry can serve as a powerful tool in both pure and applied mathematics, ultimately bridging the gap between abstract theory and practical computation. In doing so, the project not only drove forward fundamental research but also fostered the growth of a vibrant mathematical community through education, international collaboration, and strategic outreach, positioning the University of Tromsø as a leading hub for innovative mathematical research in Norway and beyond. A more detailed insight is given in the attached report.

III. PROJECT ACCOMPLISHMENTS

The project “Symmetry in Quantitative and Algorithmic Real Algebraic Geometry” (SymRAG), initiated in March 2018 under the auspices of the Tromsø Research Foundation and as part of the Pure Mathematics in Norway initiative, has achieved a broad spectrum of accomplishments over its two-phase duration. At its core, SymRAG has significantly advanced the understanding and exploitation of symmetries in mathematical computations, resulting in an impressive output of 39 research papers authored by the principal investigator, postdocs, and PhD students. The project has also made substantial contributions to education and training by directly supporting 3 postdoctoral fellows and 1 PhD student, while related initiatives have engaged an additional 5 PhD students, 2 postdocs, 8 master’s students, and 2 bachelor’s students. In its efforts to bolster long-term research in computational algebra at UiT, the team successfully secured approximately 14 MNOK in additional third-party funding. Moreover, the organization of international academic events—including 4 international conferences, 3 summer schools, and several smaller meetings—has not only facilitated

the exchange of cutting-edge ideas but also raised UiT's profile as a premier center for mathematical research in Norway and beyond. Finally, the project's extensive outreach activities and the dedicated involvement of the project manager have further cemented its impact on the broader Norwegian mathematical community. A more detailed insight is given in the attached report.

IV. EVALUATION, ASSESSMENT, AND ANNOUNCEMENT OF POSITION

The project had been evaluated in 2021, and the position of the PM has been converted to a permanent one afterwards

V. PROJECT CHALLENGES AND LESSONS LEARNED

The biggest challenges with the project came from unforeseeable issues that arose. In particular the Covid 19 pandemic had some effect on the international connections envisioned in this project and also resulted in some delays of the PhD student hire.

VI. FEEDBACK TO TFS AND UoT

The best part of the funding through TFS has been the relative high flexibility which was allowing me as a project manager to adjust the scope of the project, whenever needed. I believe that this flexibility has been one of the key factors allowing for the success of the project.

VII. CONFIRMATION

Principal investigator	*Host department / Institute leader*
Name: Cordian Riener	*Name: Martin Rypdal*
Date: February 28, 2025	*Date: March 3, 2025*
☒ *I declare that the information given in this form is correct and approved by me.*	☒ *I declare that the information given in this form is correct and approved by me.*

TROMSØ FORSKNINGSSTIFTELSE

VIII. LIST OF ATTACHMENTS/APPENDICES	YES	NO
A final accounting report from the entire project is to be submitted together with the final report.	☒	☐
List of scientific publications from the project (where the foundation is credited/acknowledged)	☒	☐
Is the text regarding the project and the PI on the TFS-website "*up-to-date".* If NO – please enclose revised text (Norwegian and/or English)	☒	☐
If the agreement requires external evaluation, enclose evaluation report according to specifications.	☐	☒
Other – specify: Concluding evaluation report	☒	☐

Symmetry in Quantitative and Algorithmic Real Algebraic Geometry

TROMSØ RESEARCH FOUNDATION

Concluding Evaluation Report of the Project, February 2025

This overview introduces the main results of the project

"Symmetry in Quantitative and Algorithmic Real Algebraic Geometry" (SymRAG).

which began in March 2018 and was organized in two parts. The project arose from an initiative of the Tromso Research Foundation (TFS) to support research and competence development in the field of pure mathematics at UiT. It was part of the Pure Mathematics in Norway project which is aimed to strengthen research in pure mathematics. The initial time frame of the project was four years with the possibility of a two year extension. While an earlier self-evaluation report from 2020 details the 33 months of the project, this document focuses on the second phase of the project.

At its core, the project explored and exploited symmetries in mathematical computations, leading to a series of 37 research papers by the project's team—including the principal investigator, postdocs, and PhD students. More detailed overview a on these research achievements are presented in the **Research** section. A significant aspect of the project was its contribution to **education and training**: Direct project involvement supported 3 postdocs and 1 PhD student, while related initiatives — partially co-financed by the project —engaged an additional 5 PhD students, 2 postdocs, 8 master's students, and 2 bachelor's students. The impact on training and development is discussed further in the **Education** section. Another key goal was to strengthen long-term research in computational algebra at UiT The Arctic University of Norway. In pursuit of this objective, the team submitted several proposals for third-party funding, successfully raising approximately additional 14 MNOK. For a deeper insight into these initiatives, please see the **Third-Party Funding** section. The project also actively organized international academic events, including **4 international conferences**, **3 summer schools**, and **several smaller regular meetings**. An overview of these events can be found in *Section 4*.

Finally, the report concludes with a summary of outreach activities and highlights the project manager's commitment to advancing the University of Tromsø and the broader Norwegian mathematical community.

1. Research

Symmetry, in its most fundamental mathematical sense, refers to the invariance of an object or a system under the actions of a group. In other words, if a set of operations (such as rotations, reflections, or permutations) leaves an object unchanged, we say that object is symmetric with respect to those operations. This simple yet powerful concept serves as the cornerstone of our project, "Symmetry in Qualitative and Algorithmic Questions in Real Algebraic Geometry," with the aim to unravel the role and impact of symmetries in algorithmic and quantitative problems.

At the heart of the project was the idea that exploiting symmetries can offer deeper insights into algebraic structures, especially those governed by polynomial systems. Many algebraic structures become significantly more transparent when analyzed through the lens of symmetry. A focus of our work has been on "**symmetry in algebraic structures**", where additional symmetries are used to better understand and simplify complex algebraic structures, such as polynomial equation systems. This perspective has led to several novel approaches in tackling longstanding problems in real algebraic geometry.

One of the key contributions in this area has been the introduction and in-depth study of special ideals known as **Specht ideals** [9, 26]. These ideals were examined within the framework of both the symmetric group S_n and the hyperoctahedral group. Alongside this, our investigations into the structure of the real orbit space [19, 34] have further enriched the understanding of how symmetries

dictate the behavior of algebraic varieties. The results emerging from these studies have not only advanced theoretical knowledge but have also found applications across various levels of mathematical research.

In addition to these algebraic investigations, our collaboration with Saugata Basu on **symmetries in homological algebra** [4, 5] has broadened the scope of our project. This series of works delves into how symmetry considerations can influence the homological properties of algebraic objects, offering new avenues for both qualitative and quantitative analysis.

The symmetric group S_n itself has proven to be a particularly fertile ground for exploring these ideas. It has been especially intriguing to extend these symmetry considerations to contexts involving "infinitely many variables." A recent breakthrough in collaboration with Mario Kummer [21] serves as a prominent example of this approach. Not only did this result push the boundaries of our theoretical understanding of geometry up to symmetry in an infinite dimensional setup, but it also opened up a host of new questions, setting the stage for further inquiry in this direction.

Beyond these theoretical pursuits, a significant thematic strand of our project has been the conceptualization of "**symmetry as a functor in algorithmic computation**". Much like a classical composition by Bach — where simple, recurring motifs undergo transformation to yield a rich and complex whole — symmetry can be harnessed to transform seemingly intractable computational problems into more manageable forms. By building on the insights gathered from the study of algebraic symmetries, we have developed several new algorithms that leverage these invariances to achieve drastic simplifications in computation [5, 8, 10, 14, 23, 33, 39]. Among these contributions, the collaborative work with Schabert and Vu [37] stands out, not only for its innovative methodology but also for earning PhD student Schabert the best PhD paper award at ISSAC 2024.

However, our work has also demonstrated that symmetry can also create computational problems of imense complexity that remain computational challenging. In particular, the investigation with Miklós [25] has revealed that the symmetries inherent in the so-called "**Immanants**" can be a double-edged sword. While symmetry can often be exploited to simplify calculations, in the case of Immanants these very symmetries are partly responsible for their computational complexity. This finding underscores a delicate balance: symmetry can lead to algorithmic simplifications in some contexts, while in others, it highlights inherent computational hardness.

A final thematic area of our research in the context of symmetry has been the **practical exploitation of symmetry in concrete computational problems**, notably in the field of optimization. Here, we have pioneered new techniques for symmetry exploitation (for example [18, 27] that have yielded fresh insights and results on several open questions (for example [12, 17]) The methodologies developed not only enhance our theoretical understanding but also have the potential to transform practical approaches in optimization and related computational domains.

Overall, the project "Symmetry in Qualitative and Algorithmic Questions in Real Algebraic Geometry" illustrates that symmetries are far more than mere aesthetic properties; they are pivotal factors that influence both the theory and practice of computational mathematics. By systematically exploring how symmetry can be harnessed — from abstract algebraic structures to concrete algorithmic implementations — we have opened up new pathways for research and innovation in real algebraic geometry and beyond.

In addition to its primary focus on symmetry in algebraic structures and algorithmic computations, the project also explored several other selected areas of research that have provided important insights across various domains of mathematics and its applications. These selected investigations —ranging from the refinement of quadrature rules and the geometric analysis of algebraic curves to the exploration of educational outcomes in mathematics —illustrate the rich diversity of research undertaken within the project.

One significant strand of investigation concerned the development of quadrature rules for integration. Integration is a classical problem in analysis: given a measure and a function, one is often interested in computing the integral of the function with respect to that measure. However, exact integration can be extremely challenging, particularly in high dimensions or complex settings. Quadrature rules offer a practical alternative by approximating the integral as a weighted sum of function evaluations at carefully chosen points, known as nodes. By fixing a predetermined level of approximation quality, a natural question arises: how many nodes are sufficient to achieve this quality, and

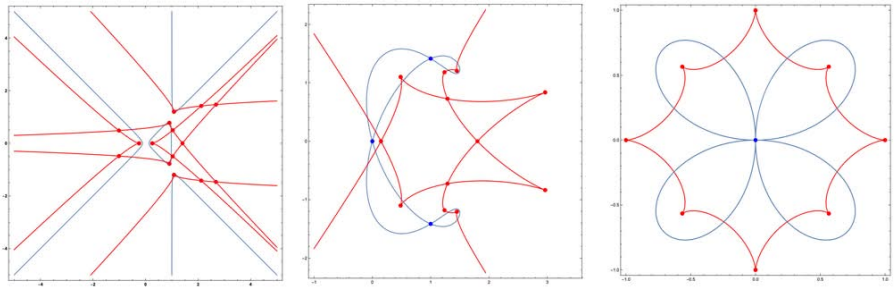

FIGURE 1. Examples of Evolutes in [32]

what is the optimal choice for their placement? A series of published works within the project has addressed these questions, developing new theoretical insights and practical algorithms. Moreover, current work in progress—conducted in collaboration with Ettore Turrati — aims to extend these ideas to the context of rational curves, thereby deepening our understanding of integration in more specialized geometric settings.

Another area of research involved the study of algebraic curves in the plane, with a particular focus on the **Evolute** of these curves. The Evolute of a curve can be thought of as the collection of all centers of curvature of the original curve. In simple terms, if you imagine a curve as a path drawn on paper, the evolute captures the locations around which the curve bends most sharply. This fascinating curve with a history going back at least to Apollonius (200 BC) offers a powerful tool for understanding the intrinsic properties of curves, such as how they change direction and how their curvature behaves. In our project, the work with Boris Shapiro and Ragni Piene [32] on the real Evolute has added a new chapter to the investigation of these curves and provided fresh perspectives on how the evolute relates to other geometric invariants, ultimately enriching the broader theory of algebraic curves.

The project also ventured into the field of mathematics education through the Learning and Understanding of Mathematics (LAUMA) initiative. The research component focused on exploring how students' high school mathematics backgrounds influence their performance and attitudes in university-level mathematics courses. The study produced in LAUMA group initiated through this project investigated several key hypotheses: it was anticipated that students who had attended higher-level mathematics courses in high school would not only exhibit more positive attitudes toward mathematics but also achieve higher grades in math-related subjects at university, compared to those who had only taken basic courses. Furthermore, the study examined differences between STEM and non-STEM students, predicting that the advanced high school background would play a similar role in both groups.

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2. Education

Mathematical research and teaching are inherently intertwined. I firmly believe that effective university teaching is rooted in active research, and conversely, research thrives when it maintains strong connections to teaching. Following this idead, the project not only advanced fundamental research in real algebraic geometry but also played a crucial role in nurturing the next generation of mathematicians.

The project contributed to education on several levels, beginning with its direct support for early-career researchers. The Postdoctoral fellows Philippe Moustrou, Thi Xuan Vu, and Ettore Teixeira Turatti benefited from financial support and direct employment through the project. Their contributions and development have been significant: Moustrou and Vu have since secured permanent positions as associate professors — Moustrou in Toulouse and Vu in Lille — while Turatti will advance to a prestigious postdoctoral position at the Max Planck Institute for Mathematics in the Sciences. In this way, the project has served as a pivotal step in the promising careers of these talented young mathematicians.

T. X. Vu

P. Noustrou

E. T. Turatti

Figure 2. The three postdoctoral members.

In addition, the project directly employed PhD student Robin Schabert, who successfully defended his thesis in the autumn of last year and now continues his academic journey at the University of Konstanz. Beyond these direct appointments, the project has been instrumental in fostering the education of junior researchers through external funding initiatives that originated from our collaborative work. For instance, the PhD students Arne Lien and Sebastian Debus have held positions financed via the POEMA network, with additional project support. Recently two new PhD students started their involvement in Tromsø within the TENORS network. Likewise, Tobias Boege joined as a postdoctoral fellow through a Marie Curie project, Rodolfo Ríos-Zertuche as a postdoctoral researcher via the MASCOT project, and Örs Rebák as a PhD student supported by departmental funding. These diverse funding streams underscore the broad impact of the project in building a strong, interconnected research community.

Furthermore, the project's principal investigator has played an active role in teaching and mentoring by supervising eight Master's students and two Bachelor's students, and by teaching several

Dr Sebastian Debus | Dr Arne Lien and Dr Robin Schabert

FIGURE 3. The three postdoctoral members.

courses. Among these, the "Lie-Størmer Master class" is particularly noteworthy. This course, directly connected to the project and co-financed by it, exemplifies how research projects can enrich the educational experience by directly linking cutting-edge research with teaching activities.

Overall, the educational efforts within the project highlight a dynamic synergy: active research informs high-quality teaching, and robust educational initiatives foster future research breakthroughs. This mutually reinforcing relationship not only advances the field of mathematics but also ensures the continuous development of its scholars, thereby securing a bright future for both research and education.

3. Additional third-part funding

The initial funding provided by TFS served as a crucial foundation for expanding the project's scope and impact. Building on this support, the principal investigator actively sought external funding opportunities, which resulted in several notable successes that not only enhanced the project's financial resources but also significantly strengthened its research and educational components.

Among the successful initiatives were two Marie Skłodowska-Curie Innovative Training Networks — POEMA and TENORS — combined providing roughly 10 MNOK. These networks were instrumental in reinforcing the project, particularly by supporting four additional PhD students, thereby expanding the team and fostering further research developments. In addition to these major funding streams, a successful application to the Norwegian Research Council for a joint project with Germany brought in a total of 100 kNOK. Although modest in comparison, this grant played an important role in advancing the research efforts within the project.

Further contributions include securing a Marie Curie fellowship for Tobias Boege and a recent award from KDir for a collaborative project with partners in India valued at 1.5 MNOK. The latter initiative aims to facilitate the exchange of PhD students, staff, and promote joint educational

FIGURE 4. The Mørketidens Mattemøte 2025

activities. Each of these individual funding successes not only exceeded the original TFS contribution in monetary terms but also significantly broadened the project's collaborative and strategic horizons.

Perhaps even more valuable than the additional financial support was the process of grant writing and the strategic insight it provided. Even the non-successful application to the Norwegian SFF scheme for the establishment of the "Lie-Størmer Center" proved beneficial. The enthusiasm and collaborative momentum generated during the application process were so impactful that the Lie-Størmer Center ultimately became a reality. Currently, several grant applications, such as those for the Doctoral School "Mathesis" and the COST-Network "Alogelis," are underway, promising to further benefit mathematics in Norway if granted.

In summary, the additional third-party funding not only augmented the financial resources of the project but also catalyzed strategic collaborations and capacity-building efforts, thereby ensuring that the initial TFS grant had a lasting and far-reaching impact.

4. Organization of Scientific Events

A central pillar of the project was the organization of a diverse range of scientific events that not only fostered academic collaboration and innovation but also significantly raised the profile of the University of Tromsø (UiT) as an important hub for mathematics both in Norway and internationally.

We organized **three summer schools** - in 2018, 2022, and 2024 — providing immersive learning environments where early-career researchers and established experts could delve into contemporary topics in real algebraic geometry and algorithmic methods. In 2019, the medium-sized conference

Our commitment to high-caliber scientific exchange was further demonstrated through several prestigious meetings. Two sessions at the renowned Oberwolfach Center for Mathematical Research in 2021 and 2025, as well as a dedicated meeting on "Symmetry, Stability, and interactions with Computation" at CIRM in France, contributed to vibrant international dialogues. The Nordic Number Theory Network Meeting organized in Tromsø in 2024 further enriched the spectrum of interdisciplinary interactions.

On the international front, the project organized three major conferences:

- **NORCOM** (Nordic Combinatorial Conference) 2022, spotlighting the latest developments in combinatorial mathematics.
- **MEGA**: Effective Methods in Algebraic Geometry 2021 (conducted online), which provided a global platform for discussing advanced computational techniques.
- **ISSAC** (International Symposium on Symbolic and Algebraic Computation) 2023, an influential event that brought together top researchers from around the world.

Additionally, the Lie-Størmer Colloquium series, held in 2022, 2023, and 2024, was specifically designed to bridge pure and computational mathematics. This series connected leading researchers

and culminated in the 2024 Tromsø edition with a special focus on symmetry, marking a significant highlight of the project. Furthermore, the annual Mørketidens Mattemøte, initiated in 2019, has

The ISSAC 2024 conference

evolved into a well-recognized event that unites mathematicians from across Norway working in various areas of applied algebra, thereby strengthening vital interdisciplinary connections.

These events played a pivotal role in showcasing UiT as a vibrant center for mathematical research. By drawing international participants and fostering high-level academic discussions, they have significantly enhanced the university's reputation and cemented its status as a key node in both national and global mathematical networks. This successful impact underscores the strategic value of the project in advancing both research and collaboration within the mathematical community.

5. Outreach and Other Responsibilities

Beyond its core research and educational initiatives, the project has significantly enhanced the public perception of mathematics and fostered a wider appreciation for the subject. The project's outreach efforts have been multifaceted and impactful.

For example, the project actively participated in the Baltic Way mathematics competition held in Tromsø, engaging young talent and showcasing the beauty of mathematical problem-solving. Additionally, sessions at the Science Center were organized to make mathematics fun and accessible for children, helping to spark early interest in the subject. The project was also present at Forskningsdage and played an active role in Skolelabben, further emphasizing its commitment to connecting with the broader community. Looking ahead, we are planning to launch a Mathematics Weekend Program in collaboration with the International School. This initiative will be open to all pupils, aiming to provide a stimulating and inclusive environment where students can explore mathematics outside of the traditional classroom setting.

In parallel to these outreach activities, the project manager has played a significant role in shaping the governance and strategic direction of mathematics at UiT and within Norway. Serving two terms on the faculty board and four years as Prodean for PhD education, he has been a driving force in academic leadership. His contributions extend to national organizations as well—having served five years on the board of the Norwegian Mathematical Society, contributed to the board of the Abel Prize, and currently leading the board of the Norwegian Mathematics Council. These roles underscore the project's broader influence, not only in advancing mathematical research but also in steering the future of mathematical education and policy in Norway.

The winning teams at Baltic Way

6. Final Remarks

This project has provided me with a unique opportunity to establish my scientific career in one of the most beautiful places on earth — Tromsø. It has been instrumental not only in directly financing my position and those of my collaborators but also in opening doors to additional funding opportunities. Most importantly, the flexible management and considerable autonomy in utilizing the funds have truly been game-changing, enabling innovative contributions that might not have been possible otherwise.

Many of the advancements that I consider pivotal to the progress of science in Norway owe their existence to this project and the support provided by TFS. I am therefore extremely grateful for this exceptional opportunity.

FIGURE 5. Culminating event: The Lie-Størmer Colloquium 2024